PREDICTIVE ANALYTICS GROUP

Essential Services Commission

Local Government – Measuring Productivity Using a Direct Method

A Comparison of DEA and Bayesian SFA

Final Report December 2017

Inherent Limitations

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In the course of our work, forecasts and/or simulations have been prepared on the basis of assumptions and methodology which have been described in our report. It is possible that some of the assumptions underlying our forecasts and/or simulations may not materialise. Nevertheless, we have applied our professional judgement in making these assumptions, such that they constitute an understandable basis for estimates and projections. Accordingly, readers of this Report must appreciate that, to the extent that certain assumptions do not materialise, our estimates and projections may vary.

CONTENTS

Glossary

Constant returns to scale (CRS)	The assumption that the relationship between inputs and outputs is constant. Namely, an increase in inputs results in commensurate and equal change in outputs.
Efficiency	Degree to which the observed use of resources to produce outputs of a given quality matches the optimal use of resources to produce outputs of a given quality.
Input oriented	Type of DEA. An input oriented DEA assumes that entities only have control over the amount of inputs and not the amount of outputs.
Output oriented	Type of DEA. An output oriented DEA assumes that entities only have control over the amount of outputs and not the amount of inputs.
Production frontier	The line or curve plotting the minimum amount of an input (or combination of inputs) required to produce a given quantity of output (or combination of outputs).
Productivity	Measure of the physical output produced from the use of a given quantity of inputs. This may include all inputs and outputs (Total Factor Productivity) or a subset of inputs and outputs (Partial Productivity). Productivity varies as a result of differences in technological change and differences in technical efficiency.
Returns to scale	Relationship between outputs and inputs. Returns can be constant, increasing or decreasing depending on whether output increases in proportion to, more or less than inputs, respectively. In the case of multiple inputs and outputs, this refers to how outputs change when there is an equi-proportionate change in all inputs.
Scale efficiency	The extent to which an entity can take advantage of returns to scale by altering its size towards optimal scale (which is defined as the region in which there are constant returns to scale in the relationship between inputs and outputs).
Technical efficiency	Conversion of inputs into outputs. Technical efficiency is determined by the difference between the observed ratio of

	combined quantities of an entity's output to input ratio achieved by best practice. It can be expressed as the potential to increase quantities of outputs from given quantities of inputs, or the potential to reduce quantities of inputs used in producing given quantities of outputs.
Technological change	The expansion or contraction of efficiency due to technological changes (i.e. the adoption of new technologies resulting in the expansion or contraction of the production frontier). In essence, this variable indicates how innovative an entity has been with their technology.
Variable returns to scale (VRS)	The assumption that the relationship between inputs and outputs is an increasing or decreasing one.

1 Overview

1.1 Background and Scope

As part of a broader program of works relating to the *Implementing a Fair Go Rates System,* the Essential Services Commission (**the Commission**) engaged Predictive Analytics Group (**PAG**) in December 2016 to measure the productivity of local governments in Victoria and then use the efficiency scores to compute efficiency factors for the Commission to consider. Guided by similar studies undertaken in other jurisdictions across Australia and the academic literature, PAG employed a quantitative method know as Data Envelopment Analysis (**DEA**) to measure productivity.

DEA is widely considered a robust and popular method for measuring the relative performance of organisations, in this case Local Governments, involved in the provision of similar (or the same) services. According to DEA, an efficient organisation is one that uses the lowest amount of inputs to provide a given amount of outputs (contingent on quality) in the context of DEA. *Total factor productivity (TFP)* is also employed to assess the change in efficiency of local governments from year to year.

Feedback was sought from the local governments and peak bodies on the findings outlined in the original report released in September 2017. The feedback was focused on the model specifications in particular, the inputs and outputs used, the lack of any use of community satisfaction data in the quantitative modelling and the development of service level DEA models that use the services provided by councils as the model outputs. Further, the quantitative framework was also criticised for its heavy reliance on DEA. In light of this criticism, this report applies an alternative method to DEA, namely Bayesian Stochastic Frontier Analysis. Stochastic frontier models such as those promulgated by Aigner, Lovell and Smidt (1977), Meeusen and van den Broeck (1977)) are the primary competitor to DEA, in studies of *firm* efficiency; see Bauer (1990) for further details.

This report applies the alternative methodology and compares the results to the original model.

1.2 Quantitative Methods

To facilitate our analysis we apply two quantitative methods, namely:

1. Bayesian SFA with Cobb-Douglas Function; and
2. Bayesian Multiple Output SFA.

Both methods are briefly discussed below.

The first method applies a Bayesian estimation methodology which is selected for its ability to address the random effect in a straightforward manner. The production frontier is specified using the Cobb - Douglas production function due to its log-linear nature and resulting computational ease. Bayesian regression theory is used to allow for sampling while ignoring economic regularity restrictions. However to enforce the economic regularity an independent Metropolis Hastings algorithm is implemented.

Method two differs slightly in that algorithms have been developed and employed to estimate multi-output production frontiers. This approach measures municipality specific efficiency relative to such frontiers. This framework is useful for comparison as it represents a departure from the existing literature that either adopts a classical econometric perspective (with restrictive functional form assumptions) or a non-stochastic approach (which directly estimates the output distance function). A Markov chain Monte Carlo algorithm has been used to implement the Bayesian inference.

Appendix A contains full details and the detailed results from each approach.

1.3 Conclusions and Recommendations

Our firm conclusion is that the DEA methodology as presented and detailed in our original report is far superior to the alternative approaches outlined in this paper. The reason is twofold:

1. Firstly, there is no statistical difference in the results, i.e. original and revised DEA methods compared with Bayesian Methods 1 and 2.
2. Secondly, the application of the two methods presented in this paper which are anchored in Bayesian estimation techniques, produce results (given the available dataset) that cannot be relied upon in a statistical sense given that the residuals are highly skewed. This indicates that the measured frontier and efficiencies of each municipality maybe spurious according to the two quantitative approaches outlined in this paper.

Appendix A - Results and Analysis

Introduction

The purpose of this section is to provide a brief overview of the results when the aforementioned methods are applied to Victorian local government data.

Method 1 – Bayesian SFA with Cobb-Douglas Function

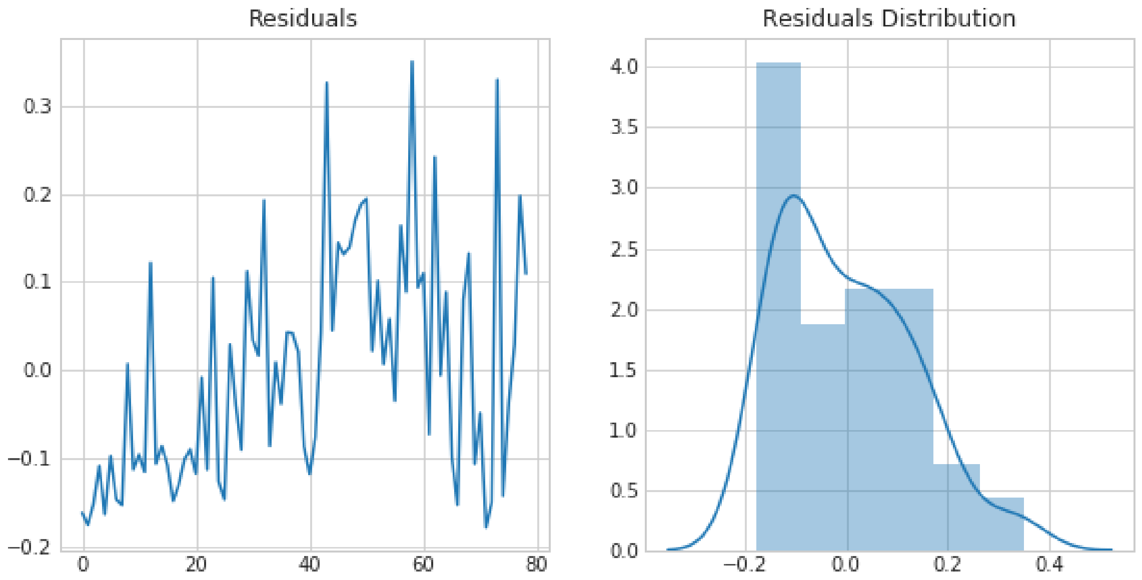
The approach employed in this analysis (refer to *Method 1 - Quantitative Approach – Technical Overview* above) requires one output variable. To produce the output variable we assumed an equal weighting in combining the number of households with the number of businesses and length of roads (measured in kilometres). Note, that this may not be a reasonable aggregation, as we are associating an equal value with each of these.

The stochastic frontier model with the Cobb-Douglas production function was used to analyse the municipality data set.

Parameter	Posterior Mean	95% CI LB	95% CI UB	IF
β_0	-5.95	-6.93	-4.88	4.25
β_1	0.20	0.0647	0.341	3.18
β_2	0.97	0.833	1.1	2.89
σ	0.18	0.119	0.243	11.9
λ	0.20	0.128	0.275	10.4

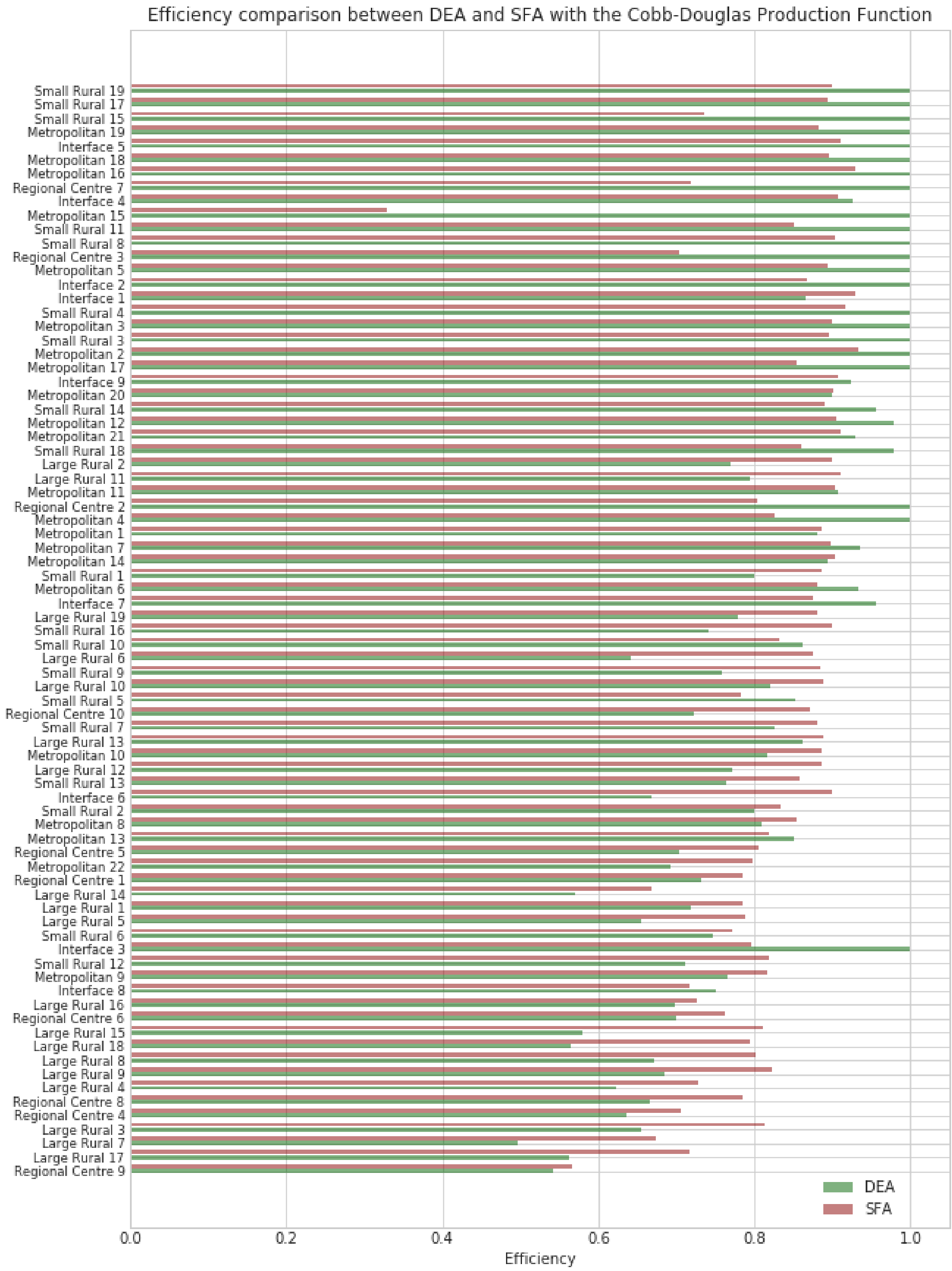
The table above reports the parameter estimates from the SFA model. The estimation used 100,000 iterations. Here the posterior mean, 95% credible interval and the so-called inefficiency factor is reported. The inefficiency factor is a measure of the simulation efficiency of the MCMC sampling scheme, and the results above suggest that the implemented sampling scheme is very efficient.

Figure 1 - Residuals



The figure above illustrates the residuals (on the left) and the residual distribution (on the right) of the SFA. The residual distribution is clearly non-normal and highly skewed. This suggests that the assumptions of equal weighting and/or the specified Cobb-Douglas production function are not reasonable, and that the data requires a more sophisticated specification. Note, in comparison the DEA method is non-parameteric, and as such is arguably robust to this kind of miss-specification.

Figure 2 - Council Performance between Models (DEA and Bayesian SFA)

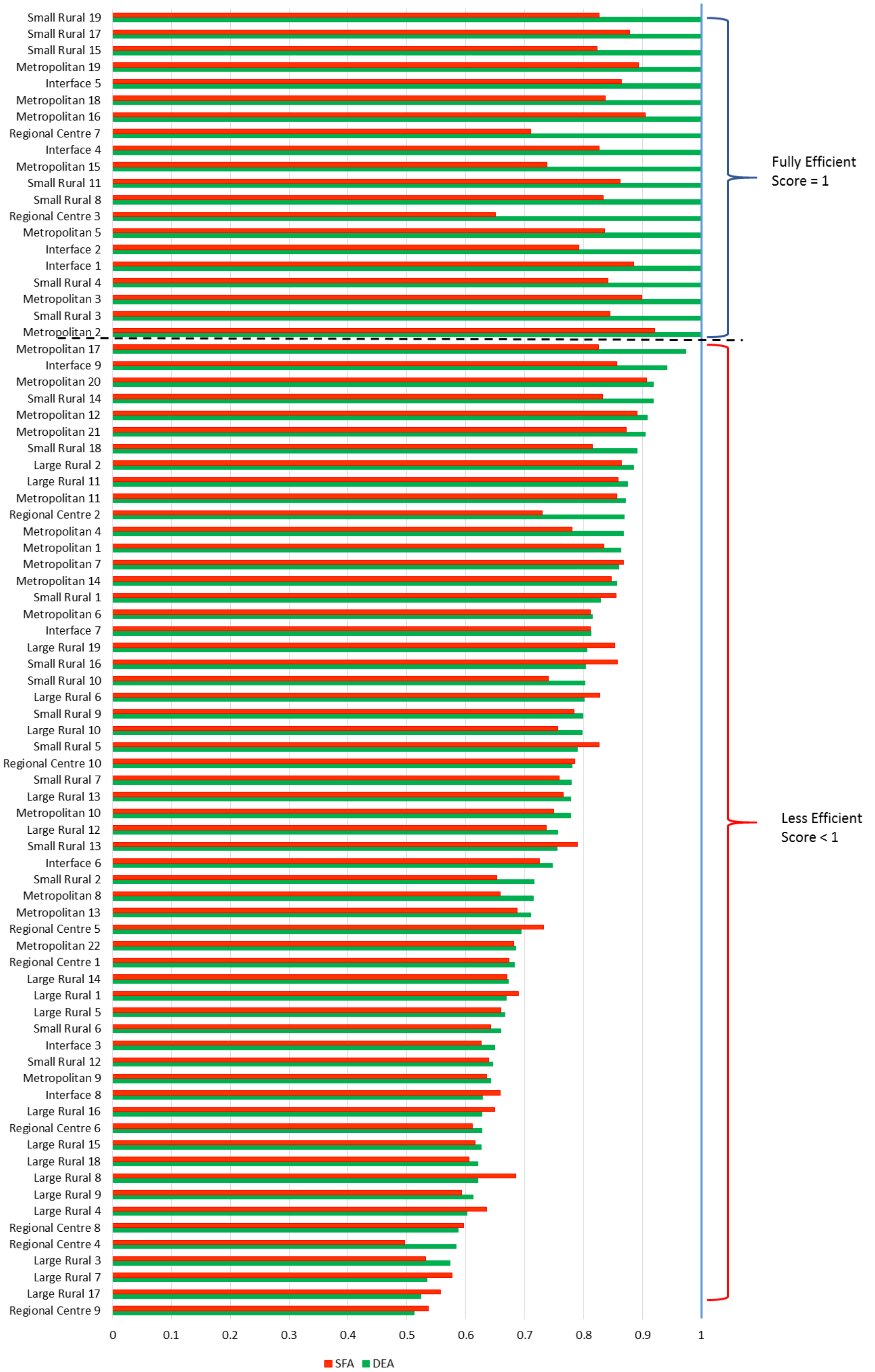


The figure above contrasts the efficiency output from both the non-parameteric DEA models and the parametric SFA model. Again, it should be re-emphasised that the results show that the assumptions underlying the SFA model do not seem reasonable. Consequently, the DEA model should be strongly preferred for this analysis.

Bayesian Method 2 – Bayesian Multiple Output SFA

As outlined earlier, the multiple output stochastic production frontier model relaxes the strict assumptions of equal weighting made under Method 1 above. The results pertaining to Method 2 – Bayesian Multiple Output SFA as presented below. According to the diagram below (Figure 6-3) none of the municipalities sit on the efficiency frontier. As such, whilst this this method entails a more complex estimation procedure, the DEA model presents more robust result. As such, in keeping with our conclusions regarding Method 1, the DEA model should once again be strongly preferred for this analysis.

Figure 3 - Council Performance between Models (DEA and Bayesian Multiple Output SFA)



Appendix B – Statistical Specification of Alternate Models

Introduction

The purpose of this section is to provide a technical overview of both methods used to facilitate the analysis.

Method 1 - Quantitative Approach – Technical Overview

Let $Y_i \in R^n$ denote the level of output from a municipality, and denote $X_i \in R^k$ as the vector of inputs, then in the stochastic frontier approach:

$$Y_i = f(X_i, \beta)\tau_i\zeta_i,$$

where $\beta \in R^k$ is a vector of unknown parameters, $\tau_i \in [0,1]$ is a measure of the municipality specific efficiency, and ζ_i is the measurement error.

The production frontier, f, is most commonly specified using the Cobb-Douglas or translog production functions. These specifications are computationally attractive, in that they are log-linear[1].

For the comparative analysis, a Cobb-Douglas specification is adopted. In particular, it is assumed that:

$$f(X_i, \beta) = \delta_0 L^{\beta_1} K^{\beta_2},$$

where L is the labour input measuring the number of person hours worked in a year, K is the capital output, δ_0 is the total factor productivity and β_1 and β_2 are output elasticities for capital and labour, respectively.

For the Cobb-Douglas production function, typically the model is linearised as follows:

$$y_i = \beta_0 + x_1\beta_1 + x_2\beta_2 - z_i + \varepsilon_i,$$

where $y_i = \log(L_1)$, $\beta_0 = \log(\delta_0)$, $x_1 = \log(L)$, $x_2 = \log(K)$, $z_i = -\log(\tau_i)$ and $\varepsilon_i = \log(\zeta_i)$.

Following Koop (2007), it is assumed that $z_i \sim G\left(1, \frac{1}{\lambda}\right)$ and that $\varepsilon_i \sim N(0, \sigma^2)$.

Here, $G(a, b)$ refers to the gamma distribution, with a mean of $\frac{a}{b}$ and a variance of $\frac{a}{b^2}$, while $N(\mu, \Omega)$ refers to the normal distribution with a mean of μ and a covariance of Ω.

Bayesian Estimation Methodology

A Bayesian estimation methodology is specified, closely following Koop (2007). A Bayesian approach is chosen because the estimation machinery developed for the Bayesian paradigm can handle the "random effect", z_i, in a relatively straightforward fashion.

[1] Log-linear refers to the econometric rather than statistical nomenclature.

Given that a Bayesian approach is being adopted, it is necessary to specify prior distributions for the hyperparameters. It is assumed that $p(\beta) \propto I(E)$, where $E = \{\beta: \beta_1 \geq 0, \beta_2 \geq 0\}$ ensures global regularity of the production frontier. Vague priors are specified for λ and τ; in particular, it is assumed that $\lambda \sim G(1, 0.01)$ and $\tau \sim G(1, 0.01)$.

A four block MCMC algorithm is used, which can be defined at iteration j as follows:

1. Sample $z^{(j)}$ from $p(z|y, \beta^{(j-1)}, \sigma^{(j-1)}, \tau^{(j-1)})$.
2. Sample $\beta^{(j)}$ from $p(\beta|y, z^{(j)}, \sigma^{(j-1)}, \tau^{(j-1)})$.
3. Sample $\sigma^{(j)}$ from $p(\sigma|y, \beta^{(j)}, z^{(j)}, \tau^{(j-1)})$.
4. Sample τ^{j} from $p(\tau|y, \beta^{(j)}, z^{(j)}, \sigma^{(j)})$.

Step 1 involves sampling from the full conditional posterior distribution of z. The form of z can be found in Koop (2007) and is derived as follows:

$$p(z|y, \beta, \sigma, \tau) \propto p(y|z, \beta, \sigma, \tau) \times p(z|\lambda)$$

where

$$\begin{aligned} p(y|z, \beta, \sigma, \tau) &\propto \exp\left(\frac{1}{2\sigma^2}(\tilde{y} + z)^T(\tilde{y} + z)\right) \\ &= \exp\left(\frac{1}{2\sigma^2}(\tilde{y}^T\tilde{y} + z^T z + 2z^T\tilde{y})\right) \end{aligned}$$

and

$$p(z|\lambda) \propto \left(\frac{1}{\lambda}\right)^n \exp\left(-\frac{\iota_n^T z}{\lambda}\right) I(0, \infty),$$

where ι_n is a vector of ones of length n. It follows that:

$$z|y, \beta, \sigma, \tau \sim N\left(X\beta - y - \sigma^2\frac{\iota_n}{\lambda}, \sigma^2 I_n\right).$$

Sampling from the posterior distribution of β while ignoring the economic regularity restrictions can be done using standard Bayesian regression theory. To enforce the economic regularity conditions an independent Metropolis Hastings algorithm is implemented (see for example, Robert and Casella (1999)).

Sampling from the posterior distributions of σ and τ in steps 3 and 4 are completely standard (see Koop (2007) for further details).

Method 2 - Quantitative Approach – Technical Overview

Following Fernandez et al. (2000), an empirical implementation using Markov chain Monte Carlo (MCMC) methods is applied. We discuss the approach below.

We note the following is of a technical nature. The information is important to allow third parties to replicate our model and results.

We begin with a description of the production technology, which is represented mathematically via the following transformation function:

$$f(y, x) = 0$$

where y is a vector of p outputs and x is a vector of inputs.

A large body of literature (e.g. Färe and Primont, 1990) exists in which this function is implicitly estimated to allow for the evaluation of firm specific productivity using non-econometric approaches. This approach relies on linear programming techniques and assumes a deterministic transformation function (i.e. no measurement error in the data). In many instances, this approach is completely reasonable. However, if the dataset is noisy, it can be more appropriate to adopt an econometric approach and formally model measurement error (further discussion may be found in Koop et al. (1997, 1999)).

The survey undertaken by Fernandez et al. (2000) however, only located three examples of prior research in which the economic approach was adopted (consider Adams et al. (1996, 1999) and Lothgren (1997)). Additionally, in all three cases the separability in the transformation function was assumed. To facilitate the current analysis, a constant elasticity of transformation form is assumed as is the multivariate character of the data through the specification of a p-dimensional sampling model.

The method used is Bayesian because as discussed in the work of Koop et al. (1997, 1999), it allows for the calculation of exact finite sample properties of all features of interest (including municipality-specific efficiency), and overcomes some problematic statistical issues associated with the classical estimation of stochastic frontier models. This is not to suggest that the Bayesian approach is superior to the classical econometric or linear programming approaches. Different approaches have various uses and limitations and the Bayesian method should be regarded as another technique when working with multiple output and production frontier models.

Model Specification

The model considers a set of NT observations corresponding to outputs of N different municipalities over T time periods. The output of municipality i $(i = 1, \dots, N)$ at time t $(t = 1, \dots, T)$ is p-dimensional and given by the vector $y_{(i,t)} = (y_{(i,t,1)}, \dots, y_{(i,t,p)})' \in \Re_+^p$.

The stochastic frontier model with composed error is extended from the case of a single output to the case of multiple outputs by applying the following transformation to the p-dimensional output vector:

$$\theta_{(i,t)} = \left(\sum_{j=1}^{p} \alpha_j^q y_{(i,t,j)}^q \right)^{1/q}$$

where $\alpha_j \in (0,1)$ for all $j = 1, \dots, p$ and such that $\sum_{j=1}^{p} \alpha_j = 1$and with $q > 1$. For fixed values of $\alpha = (\alpha_1, \dots, \alpha_p)'$, q and $\theta_{(i,t)}$, $\theta_{(i,t)}$ described above defines a $(p-1)$ dimensional surface in $\Re_+^p$ corresponding to all the p-dimensional vectors of outputs $y_{(i,t)}$, that are technologically equivalent (i.e. $\theta_{(i,t)}$ plots the production equivalence surface).

Technological efficiency is captured by the fact municipalities may lie below the frontier, resulting in a vector of inefficiencies $\gamma \equiv D_Z \in \Re_+^{NT}$, where D is an exogenous $NT \times M$ $(M \leq NT)$ matrix and $z \in \mathcal{L}$ with $\mathcal{L} = \{z = (z_1, \dots, z_M)' \in \Re^M : Dz \in \Re_+^{NT}$. Different choices of D, allow for the accommodation of various amounts of structure on the vector γ of inefficiencies. As an example, if $D = I_{NT}$, the NT- dimensional identity matrix leads to an inefficiency term which is specific to each different municipality and time period. $D = I_N \otimes \iota_T$, where $\iota_T$is a T-dimensional vector of ones and $\otimes$ denotes the Kronecker product, implies inefficiency terms which are specific to each municipality, but constant over time (i.e. 'individual effects'). As the model is working in terms of δ, the log of the aggregate output, the efficiency corresponding to municipality i and period t will be defined as $\exp(-\gamma_{(i,t)})$ where $\gamma_{(i,t)}$ is the appropriate element of γ.

Appendix C - Bibliography

Relevant references mentioned in the preceding report are:

1. Aigner, D. Lovell, C. and Schmidt, P. (1977), “Formulation and estimation of stochastic frontier production function models”, Journal of Econometrics.
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